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Time development of coherent and superfluid properties in the course of a λ-transition

E Levich and V Yakhot

Department of Nuclear Physics, Weizmann Institute of Science, Rehovot, Israel
Departments of Structural Chemistry and Chemical Physics, Weizmann Institute of Science, Rehovot, Israel

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Abstract. The microscopic theory of the time evolution of an interacting Bose gas being cooled to the temperature $T < T_c$ is presented. The considerations are based on the exact (in classical limit) equations of motion derived by J S Langer. It is shown that at the initial stages of relaxation the sharp peak appears at the low energy tail of the distribution function. The following time evolution of this peak leads to a δ function formation. When this peak is too narrow the Boltzmann equation we use is no longer applicable and one must apply the exact equations of motion or the quasi-Boltzmann equation derived in this work which takes into account the coherent effects. The evolution of the distribution function and of the energy spectrum in the system is given by the solution of the equation of motion.

The development of a new phase is of explosive character and has a lot of features in common with instabilities in non-linear plasmas, optics, etc, and thus is of quite a general nature. In addition it is shown that the phenomenological models widely used for the description of the dynamics of the λ-transition in a critical region can be derived from microscopic equations of motion.

1. Introduction

In this paper we present a microscopic theory of the non-linear dynamics of phase transformations (e.g., the λ-transition in helium) with the smooth (in time) appearance of superfluidity in the system. This work is based on exact (in classical limit) microscopic equations of motion for bosons derived by Langer (1968–1969) in the coherent states representation. The general concepts developed in our work may also be applied to a wide class of second-order phase transitions, such as occur in quantum spin systems. This is true, since it is possible to treat the quantum spin system phase transitions as a generalised Bose condensation. Rigorous proof of the last statement is given by Goldhirsch *et al.* (1978).

Remarkable progress has been achieved during the last decade in the theory of critical dynamics. The bulk of the work has been based, so far, upon two related phenomenological approaches, mode–mode coupling theory and Onsager-type irreversible equations of motion, with the renormalisation group treatment subsequently applied (Halperin *et al* 1974, 1976, Kawasaki 1976, Hohenberg and Halperin 1977).

These approaches have led to the explanation of a great variety of experimental data and pointed out some general features of dynamics in systems near to the critical point. Despite these unquestionable achievements one might notice that, so far,

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phenomenological models in the theory of critical dynamics have not explained some important experimental facts, for example the behaviour of the density–density correlation function for a λ-transition. Thus it is desirable to develop a microscopic theory of critical dynamics which would support the phenomenological models and to understand the critical dynamics in more detail. Several attempts have been made, but the consistency of the microscopic and phenomenological approaches is not yet rigorously proved.

However, there is another dynamic problem of phase transitions which is only slightly touched upon by investigators and which is far from being understood. All the theories quoted above consider the response of the system in the critical region with respect to perturbations which keep the temperature unchanged.

The initial temperature of the system and the final one, when it relaxes back to equilibrium, are the same. They are both either above the critical temperature or below it. In other words, in these theories perturbation does not lead the system through the critical point. The most common feature of relaxational processes considered in the critical region is the critical slowing down: the relaxation time goes to infinity when $T \to T_c$.

One may ask several very different questions, with respect to the dynamics of phase transitions. What would happen to the system which initially is at a temperature above T_c and which is perturbed in such a way that finally it comes to equilibrium at a temperature below T_c? For example, we can instantaneously take a certain amount of heat from the system or put it into contact with a heat bath at a temperature below T_c. The trivial answer to this question is the following. In the course of relaxation to the new state of equilibrium a new phase should be developed which would mean that a phase transition has occurred. Thus the system acquires drastically new properties, such as superfluidity, etc. A much more complicated answer would follow if we were interested in the time development of this process, in a time evolution of the system which leads to the appearance of new coherent qualities in it. The attempt to answer these questions is the main subject of this work.

So far only a few papers relating to this problem have appeared. Some of them consider the time-dependent Landau–Ginzburg model (Kawasaki 1977, Kawasaki *et al* 1978) or phenomenological Fokker–Planck-type equations (Suzuki 1976, 1977)—the models without conservation laws. These works have not led to the solution for the final stage of relaxation to equilibrium. One should recall that in conventional critical dynamics the conservation laws are of utmost importance and make the dynamics very different in comparison with purely relaxational models without conservation laws. One may expect that in the case of the passage of the system through the critical point the dynamics will be different for realistic systems with conserved integrals of motion and purely relaxational models.

In another series of papers the kinetics of Bose condensation in simple models of ideal and weakly interacting Bose gases is considered (Zeldovich and Levich 1969, Coste and Peyraud 1975, Chaplin *et al* 1974, Levich and Yakhot 1977a, b). However, the Boltzmann equations used in these papers are not applicable for the final stage of the process, though they indicate the smooth (in time) appearance of coherence in the system. Besides, this treatment was valid only outside the critical region, when both initial and final temperatures were far from T_c.

In this work we extend our treatment so that it is appropriate in the critical region and at all stages of coherence that appear in the system in the course of a phase transition. It will be shown that the time evolution of a Bose system with the

subsequent appearance of superfluidity (Bose condensate) has many features in common with some general processes displaying instability, such as interacting waves in optics, non-linear plasmas, etc. Therefore, our results may be of quite a general nature.

Finally, we will show how to derive the Halperin *et al* (1976) phenomenological equations for the dynamics of helium, based on the initially purely microscopic Hamiltonian. The general scheme is valid for spin systems also. The appropriate development for the Ising and Heisenberg models will be published elsewhere.

2. The equations of motion

In this work we consider the dynamics of an interacting Bose gas in contact with a model heat bath of infinite heat capacity. We can write the Hamiltonian of the system:

$$H = H_{BB} + H_{BT} \tag{2.1}$$

$$H_{BB} = \sum \frac{\hbar K^2}{2m} \hat{a}^+_{\boldsymbol{K}} \hat{a}_{\boldsymbol{K}} + \frac{1}{V} \sum_{\boldsymbol{K}_1+\boldsymbol{K}_2=\boldsymbol{K}_3+\boldsymbol{K}_4} U^{\boldsymbol{K}_3\boldsymbol{K}_4}_{\boldsymbol{K}_1\boldsymbol{K}_2} \hat{a}^+_{\boldsymbol{K}_1} \hat{a}^+_{\boldsymbol{K}_2} \hat{a}_{\boldsymbol{K}_3} \hat{a}_{\boldsymbol{K}_4}$$

$$H_{BT} = \frac{1}{V} \sum_{\boldsymbol{K}_1+\boldsymbol{K}_2=\boldsymbol{K}_3+\boldsymbol{K}_4} V^{\boldsymbol{K}_3\boldsymbol{K}_4}_{\boldsymbol{K}_1\boldsymbol{K}_2} \hat{a}^+_{\boldsymbol{K}_1} \hat{b}^+_{\boldsymbol{K}_2} \hat{a}_{\boldsymbol{K}_3} \hat{b}_{\boldsymbol{K}_4}.$$

The operators $\hat{a}^+_{\boldsymbol{K}}$, $\hat{a}_{\boldsymbol{K}}$ and $\hat{b}^+_{\boldsymbol{K}}$, $\hat{b}_{\boldsymbol{K}}$ correspond to Bose and model heat bath particles respectively. H_{BB} describes the interaction between Bose particles, while H_{BT} corresponds to Bose gas–heat bath interactions. We assume a usual approximation for the matrix elements of interaction:

$$U^{\boldsymbol{K}_3\boldsymbol{K}_4}_{\boldsymbol{K}_1\boldsymbol{K}_2} = \int U(r) \operatorname{ecp}\left(\frac{\mathrm{i}}{\hbar}(\boldsymbol{K}_1+\boldsymbol{K}_2-\boldsymbol{K}_3-\boldsymbol{K}_4)\boldsymbol{r}\right) \mathrm{d}\boldsymbol{r} = U_0$$

$$V^{\boldsymbol{K}_3\boldsymbol{K}_4}_{\boldsymbol{K}_1\boldsymbol{K}_2} = \int V(r) \exp\left(\frac{\mathrm{i}}{\hbar}(\boldsymbol{K}_1+\boldsymbol{K}_2-\boldsymbol{K}_3-\boldsymbol{K}_4)\right) \mathrm{d}\boldsymbol{r} = V_0. \tag{2.2}$$

Let us introduce the coherent states representation (Langer, 1968, 1969, Klauder 1960) useful for the Bose systems:

$$\hat{a}|\lambda\rangle = a|\lambda\rangle; \qquad \langle\lambda|\hat{a}^+| = \langle\lambda|a^* \tag{2.3}$$

where a and a^+ are complex numbers defining the eigenstates of operators $\hat{a}$ and $\hat{a}^+$, and the wavefunction of the system

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{V}} \sum a_{\boldsymbol{K}} \mathrm{e}^{\mathrm{i}\boldsymbol{K}.\boldsymbol{r}}. \tag{2.4}$$

From the definition (2.3) one can see that to write the Hamiltonian in the coherent states representation it is necessary to substitute in the Hamiltonian the complex numbers a and a^*, instead of the operators $\hat{a}$ and $\hat{a}^+$ respectively. Then in the classical limit, $\hbar \to 0$, one can write the Hamiltonian equations of motion (Langer 1968, 1969):

$$\mathrm{i}\hbar \frac{\partial \psi}{\partial t} = \frac{\delta H(\psi, \psi^*)}{\delta \psi^*} \tag{2.5}$$

$$i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\delta H(\psi, \psi^*)}{\delta \psi} \tag{2.6}$$

where $H(\psi, \psi^*)$ is the Hamiltonian H_{BB} in the coherent states representation. Similarly we may consider $b_p^\dagger b_p$, in H_{BT} as some classical field $M(\boldsymbol{K}, \boldsymbol{K}')$ and treat the H_{BT} part of the interaction in the same way. Then we will have the following Hamiltonian equation of motion for the coherent states $a_{\boldsymbol{K}}$:

$$i\hbar \frac{\partial a_{\boldsymbol{K}}}{\partial t} = \frac{\hbar K^2}{2m} a_{\boldsymbol{K}} + \frac{U_0}{V} \sum a^*_{\boldsymbol{K}_2} a_{\boldsymbol{K}_3} a_{\boldsymbol{K}_4} + \frac{V_0}{V} \sum M(\boldsymbol{K}, \boldsymbol{K}') a_{\boldsymbol{K}'}. \tag{2.7}$$†

One may notice that the free part in equation (2.7) can be absorbed easily in the phase of $a_{\boldsymbol{K}}$, so that equation (2.7) can be rewritten

$$i\hbar \frac{\partial a_{\boldsymbol{K}}}{\partial t} = \frac{U_0}{V} \sum a^*_{\boldsymbol{K}_2} a_{\boldsymbol{K}_3} a_{\boldsymbol{K}_4} \exp[i(\omega_1 + \omega_2 - \omega_3 - \omega_4)t] + \frac{V_0}{V} \sum M(\boldsymbol{K}, \boldsymbol{K}') a_{\boldsymbol{K}'} \tag{2.8}$$

where $\omega(K) = \hbar K^2/2m$. If the coupling constants are small or the system is diluted one can easily derive the Boltzmann kinetic equation from (2.8). Assuming that the time of evolution is large so that

$$(\omega_1 + \omega_2 - \omega_3 - \omega_4)t \gg 1 \tag{2.9}$$

and phases of a, a^* are random, i.e.

$$\langle a^*_{\boldsymbol{K}_1} a_{\boldsymbol{K}_2} \rangle = |a_{\boldsymbol{K}_1}|^2 \delta(\boldsymbol{K}_1 - \boldsymbol{K}_2). \tag{2.9'}$$

One would readily derive the Boltzmann equation for the distribution function $n_{\boldsymbol{K}} = |a_{\boldsymbol{K}}|^2$ (Sagdeev 1973):

$$\frac{\partial n_{\boldsymbol{K}}}{\partial t} = I_{BB} + I_{BT} \tag{2.10}$$

$$I_{BB} = -\frac{U_0^2}{\hbar V^2} \sum \delta_{\boldsymbol{K}_1 + \boldsymbol{K}_2 = \boldsymbol{K}_3 + \boldsymbol{K}_4} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)[n_{\boldsymbol{K}_1} n_{\boldsymbol{K}_2} (n_{\boldsymbol{K}_3} + n_{\boldsymbol{K}_4})$$

$$- n_{\boldsymbol{K}_3} n_{\boldsymbol{K}_4} (n_{\boldsymbol{K}_1} + n_{\boldsymbol{K}_2})]. \tag{2.11}$$

The form of I_{BB} depends, generally speaking, on the properties of the heat bath. In a particular case when we assume the heat bath to be a Fermi gas with the distribution function $N(\boldsymbol{K})$

$$I_{BT} = -\frac{V_0^2}{\hbar V^2} \sum \delta_{\boldsymbol{K}_1 + \boldsymbol{K}_2 = \boldsymbol{K}_1 + \boldsymbol{K}_2} \delta(\omega_1 + \omega_2 - \omega_1' - \omega_2') n_{\boldsymbol{K}_1} n_{\boldsymbol{K}_1} (N_{\boldsymbol{K}_2} - N_{\boldsymbol{K}_2}). \tag{2.12}$$

One may notice that in the Boltzmann equations (2.10)–(2.12) derived from equation of motion (2.7) we have induced scattering terms only and no spontaneous terms at all. This is certainly due to the classical nature of equation (2.7). If one would like to take into account spontaneous terms as well, a slightly different method of derivation of the Boltzmann equation can be applied. We can write in a straightforward way in

† We deliberately left $\hbar$ in front of the left-hand side in (2.7) so as to give to Hamiltonian equations of motion the form of the non-linear Schrödinger equation. Such a form is convenient in various applications and does not contradict the classical nature of Hamiltonian equations.

the second-order perturbation expansion:

$$\frac{\partial n_{\boldsymbol{K}}}{\partial t} = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \frac{1}{\hbar^2} \operatorname{Re} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\epsilon t} \langle [H(t), [H, \hat{n}_{\boldsymbol{K}}]] \rangle \tag{2.13}$$

where $\hat{n}_{\boldsymbol{K}} = \hat{a}^+_{\boldsymbol{K}} \hat{a}_{\boldsymbol{K}}$. Equation (2.13) contains the time-dependent part of the interaction operator which is proportional to $\exp[\mathrm{i}(\omega_1 + \omega_2 - \omega_3 - \omega_4)t]$. If condition (2.9) is fulfilled, that is the process is slow, the phases change many times during the time of the process and thus we can again apply the random phase approximation (RPA). Subsequently we derive from (2.13) a Boltzmann equation which consists of both spontaneous and induced terms. For example for I_{BB}

$$I_{\mathrm{BB}} = -\frac{U_0^2}{\hbar V^2} \sum \delta_{\boldsymbol{K}_1 + \boldsymbol{K}_2 = \boldsymbol{K}_3 + \boldsymbol{K}_4} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times [n_{\boldsymbol{K}_1} n_{\boldsymbol{K}_2} (1 + n_{\boldsymbol{K}_3})(1 + n_{\boldsymbol{K}_4}) - n_{\boldsymbol{K}_3} n_{\boldsymbol{K}_4} (1 + n_{\boldsymbol{K}_1})(1 + n_{\boldsymbol{K}_2})] \tag{2.14}$$

$$I_{\mathrm{BT}} = \frac{V_0^2}{\hbar V^2} \sum \delta_{\boldsymbol{K}_1 + \boldsymbol{K}_2 = \boldsymbol{K}_3 + \boldsymbol{K}_4} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times [n_{\boldsymbol{K}_1} N_{\boldsymbol{K}_2} (1 + n_{\boldsymbol{K}_3})(1 - N_{\boldsymbol{K}_4}) - n_{\boldsymbol{K}_3} N_{\boldsymbol{K}_4} (1 + n_{\boldsymbol{K}_1})(1 - N_{\boldsymbol{K}_2})].$$

What would happen if condition (2.9) was not fulfilled? Then the random phase approximation is not valid any more since violation of (2.9) means an appearance of coherence in the system. With respect to the coherent system one may consider different extreme situations. Assume, for example, that our system consists of an infinite number of modes concentrating in a narrow energetic interval. In this case one can apply the opposite approximation to (2.9) for the scattering processes within this narrow interval, i.e.

$$(\omega_1 + \omega_2 - \omega_3 - \omega_4)t \ll 1. \tag{2.15}$$

Here we assume that the process is fast and the system has strong temporal coherence. If we assume that our system is isotropic and stable with respect to spatial perturbations, then we may still use the approximation of random phases but not make time $t \to \infty$, as is done in the derivation of the Boltzmann equation. It is still possible to decouple the four amplitude correlators into pair correlators, the step which is crucial in the derivation of the Boltzmann equation. The expressions for the collision term would become in such approximations

$$I_{\mathrm{BB}} = -\frac{U_0^2}{\hbar^3 V^2} \sum \frac{\sin^2[(\omega_1 + \omega_2 - \omega_3 - \omega_4)t]}{(\omega_1 + \omega_2 - \omega_3 - \omega_4)^2 t} [n(\boldsymbol{K}_1) n(\boldsymbol{K}_3)(n(\boldsymbol{K}_2) - n(\boldsymbol{K}_4))$$

$$+ n(\boldsymbol{K}_2) n(\boldsymbol{K}_4)(n(\boldsymbol{K}_1) - n(\boldsymbol{K}_3))]. \tag{2.16}$$

Instead of $\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$ in the Boltzmann equation we have in approximation (2.15), the time factor t. If one tries to solve equation (2.16), condition (2.15) should not be forgotten. It is well known from general statistical physics that the derivation of the Boltzmann equations contains an inconsistency. Namely, the dispersion relation $\omega(K)$ is kept unchanged, being that of the free particles. This is certainly a bad approximation in the cases in which the interaction is strong. It is also bad for processes described by the quasi-Boltzmann equation (2.16). Indeed this kinetic equation was derived on the assumption that the relaxation was very fast, using the

perturbation theory. In cases when perturbation is not small but acts for a short time we should have, in accordance with the uncertainty principle, an uncertainty of energy in a final state $\Delta E \simeq \hbar/t$. This may be interpreted in classical terms as a non-linear shift of the frequency due to interaction, as in the case of non-linear parametric resonance in classical mechanics. This circumstance is of the utmost importance for the problem considered in this paper, as we shall see from the development. It will be shown that at the last stages of relaxation with the new phase formation the processes are of an explosive character and the interaction, even if it was initially small, becomes very strong in the narrow range of K. This, of course, affects the dispersion relation which cannot be considered the same as it was in the system with weak interaction. The very interesting property of equation (2.16) is that it easily provides us with the time scale of the process independently of the initial shape of the distribution. Indeed with the help of the invariant number of particles, N_0, which is the integral of motion, we can introduce (only one) dimensionless time, $\tau = t(U_0 N_0/\hbar V)$, so that it is easy to conclude that the time scale of the process governed by (2.16) is

$$\tilde{\tau} \simeq \frac{\hbar}{U_0(N_0/V)}. \tag{2.17}$$

We should like to emphasise that equation (2.16) is definitely not always applicable. The coherence, in general, may create non-uniformity and non-isotropy in the system. Then condition of stochasticity may not be fulfilled (Lvov and Rubenchik 1977). In other words equation (2.16), as it was pointed out above, corresponds to the approximation when we take care of temporal coherence only, and neglect the effect of spatial coherence.†

As to the general properties of equations (2.5), (2.14), (2.16) we may state the following:

(1) All of them conserve the total number of particles.

(2) All of them conserve the total energy of the system.

(3) The equilibrium solution of (2.14) above T_c is

$$n_K = \left[\exp\left(\frac{\hbar\omega(K)+\mu}{kT}\right) - 1\right]^{-1}.$$

(4) Below T_c equation (2.5) is indeed appropriate to describe superfluidity since it is identical to the usual hydrodynamical equations for superfluid motion (Langer 1968, 1969). The spectrum of excitations which follows from (2.5) is $\omega = cK$.

(5) With respect to hydrodynamical modes in the system above T_c one may notice the following. As was mentioned above, the loss of information about the evolution of the dispersion relation due to interaction occurs in the transition from reversible Hamiltonian (equation (2.5)) to irreversible Boltzmann equation. What is usually done in such a case, is that, after the Boltzmann equation is solved, the poles of the corresponding response function (the dielectric function in plasma physics) should be looked at. They determine the diffusion spectrum, sound wave spectrum, etc. Alternatively, one can introduce into the left-hand side of the Boltzmann equation the mean-field terms (Vlasov terms in plasma physics terminology). These are the two

† *Note added in proof.* In the general case when the non-linearity in the system is large and yet the system is stochastic due either to the stochastic initial conditions or to random interactions with an external field, the situation is much more complicated and akin to the strong turbulence. One should pass over to the Dyson–Wild equations. (Zacharov and Lvov 1975).

usual ways to extract the hydrodynamical modes from the Boltzmann equation (Ma and Senbetu 1977, Sak 1976). Another way of introducing the hydrodynamics into the problem is to do it by perturbation methods using the Hamiltonian equations of motion. If one calculates the new vertex perturbatively then, after some calculation, it will consist of only two terms (Kwok and Martin 1966). One is the structureless constant of interaction U_0 and the other, with structure, having poles on the imaginary axes, and proportional to U_0^2. The position of the poles defines the diffusion part of the spectrum $\omega = iDK^2$. Subsequently, the diffusion can be attributed to the self-energy part G' of the propagator:

$$\frac{1}{G'} \propto r + \omega - \frac{\hbar^2 K^2}{2m} + iDK^2. \tag{2.18}$$

Thus the dynamic equation, corresponding to this corrected 'free' propagator is:

$$\frac{i\hbar \partial a_{\boldsymbol{K}}}{\partial t} = \left(r + \frac{K^2}{2m} + iDK^2\right) a_{\boldsymbol{K}} + \frac{1}{V} \sum U_0^* a_{\boldsymbol{K}_1}^* a_{\boldsymbol{K}_2} a_{\boldsymbol{K}_3}. \tag{2.19}$$

Im $U_0^* \neq 0$ now. The Boltzmann equation contains a new non-linear term (see § 6):

$$\frac{\partial n_{\boldsymbol{K}}}{\partial t} = (\operatorname{Im} r - DK^2)(n_{\boldsymbol{K}} - n_{\boldsymbol{K}\mathrm{eq}}) + n_{\boldsymbol{K}} \sum \operatorname{Im} U_0^* (\boldsymbol{K}, \boldsymbol{K}') n_{\boldsymbol{K}'} + I_{\mathrm{BB}} \tag{2.20}$$

where the appearance of $n_{\boldsymbol{K}\mathrm{eq}}$ is the result of the noise source which we have to introduce in order to obtain a non-zero equilibrium solution. If we are close to equilibrium then I_{BB} is not important since the role it plays is already contained in the constant Im r. In the opposite case, when we are far from equilibrium, the I_{BB} term is dominant In general when the RPA is not applicable and the system is far from equilibrium, its evolution is described by (2.19) with the dominant non-linear interaction or the equivalent Dyson–Wild equations.

What would be changed in the previous considerations if we move into the critical region? It is clear that we should modify the equations of motion to take care of equilibrium critical fluctuations which are important when the temperature is close to T_c. In order to do it we shall make use of the Halperin–Hohenberg–Siggia (HHS) phenomenological model (Halperin *et al* 1976), in a slightly different form, convenient for our purposes. This model describes helium by means of the following two dynamic equations for the order parameter ψ and the field m coupled to ψ:

$$\frac{\partial \psi}{\partial t} = \Gamma_0 \frac{\delta F}{\delta \psi^*} + ig_0 \psi \frac{\delta F}{\delta m} + \theta \tag{2.21}$$

$$\frac{\partial m}{\partial t} = \lambda_0 \nabla^2 \frac{\delta F}{\delta m} + 2g_0 \operatorname{Im}\left(\psi^* \frac{\delta F}{\delta \psi^*}\right) + \xi \tag{2.22}$$

where the free energy:

$$F = \int \mathrm{d}^d r \left(\tfrac{1}{2} r_0 |\psi|^2 + \tfrac{1}{2} |\nabla \psi|^2 + U_0 |\psi|^4 + \tfrac{1}{2} C_0^{-1} m^2 + r_0 m |\psi|^2\right) \tag{2.23}$$

and θ and ξ are the noise sources.

First of all let us show briefly that equations (2.5) and (2.6), which are the exact microscopic equations, can be used in order to derive the phenomenological model of

HHS. To do it we multiply equation (2.5) by ψ^* and equation (2.5) by ψ and find

$$\frac{\mathrm{i}\mathrm{d}|\psi|^2}{\mathrm{d}t} = -(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = -2\mathrm{i}\,\mathrm{Im}(\psi^*\nabla^2\psi). \tag{2.5'}$$

If we denote $|\psi|^2$ by m, then the equation for m is similar to (2.22) for the auxilary field coupled to ψ except for the diffusion term $\lambda\nabla^2(\psi\psi^*)$ (Halperin *et al* 1976). This term, as well as the noise cannot appear from the microscopic Hamiltonian equation of motion since they are purely dissipative in nature. However, it does appear after the first step of iteration in the RGA as one can see from equation (3.12) (Halperin *et al* 1976). Analogously, after the first step of iteration the coefficients of the linear terms in (2.5), (2.6) become complex and the hamiltonian acquires the form of the free energy functional (2.23). Subsequently equations (2.5) and (2.6), become similar to that in the HHS model. It is essential that in the course of renormalisation the density m becomes more singular than $|\psi|^2$. Consequently in a renormalised system the density of particles cannot be represented as $|\tilde{\psi}|^2$ any more, even if $m = |\psi|^2$ initially.† In the course of renormalisation equations (2.21) and (2.22) acquire very complicated forms. However, in terms of the $\epsilon = 4 - d$ expansion they keep their form invariant with the altered parameters (Halperin *et al* 1976). We do not dwell upon these complicated matters and assume that the conclusions stemming from the HHS model are correct and coincide with those which could have been obtained, in principle, from equation (2.5) without recourse to the HHS model. The way to do it is indicated by Halperin *et al* (1976) and consists of steps leading to equation (2.19) with subsequent application of the RGA. The latter should lead to the same value for kinetic coefficients as those obtained from the HHS model. The vertex will again consist of two terms in the first-order in the ϵ-expansion. One is the structureless constant of renormalized interaction proportional to ϵ and the other determines the dispersion relation for hydrodynamic modes.

Finally we come to the conclusion that the equation of motion in the critical region can be written in the same form in the first order in the ϵ-expansion as (2.19) but with the altered diffusion coefficient and interaction constant appropriate for the critical region

$$\frac{\mathrm{i}\hbar\partial\tilde{a}_{\boldsymbol{K}}}{\partial t} = (\tilde{r} + K^2 - \mathrm{i}\tilde{D}K^2)\tilde{a}_K + \frac{1}{V}\sum \tilde{U}_0\tilde{a}_{\boldsymbol{K}_1}\tilde{a}^*_{\boldsymbol{K}_2}\tilde{a}_{\boldsymbol{K}_3}. \tag{2.24}$$

$\tilde{U}_0$ now contains an imaginary part (see § 6). In the critical region equation (2.24) gives the correct relaxation time for small perturbations close to equilibrium.

It is important that, in the course of renormalisation, while deriving equation (2.24), only high-momentum modes $K > 1/\xi$ are being integrated out. These modes are considered to be close to equilibrium. On the other hand this equation is valid for long hydrodynamic modes $K < 1/\xi$ even if they are far from equilibrium, since no assumptions were made in the derivation of (2.24) as to the state of these modes. If the amplitudes of these modes are large and they are unstable, the non-linear interaction becomes dominant and the above considerations with respect to non-equilibrium evolution are still valid.

† This understanding of the inter-relation between equations (2.5), (2.6) and the HHS model was achieved with S Goldhirsch.

We should like to mention that the shortcoming of the derivation of the HHS model given above, is that we did not have noise sources in the final equations. These should evidently appear if one applies more general statistical methods (Mori 1965).

To conclude this section we would like to mention that the reversible quasi-Boltzmann equation (2.16) resembles the approach used for calculation of the contribution to the heat conductivity caused by phonon scattering on the object with the short lifetime. In this case the energy conservation per collision was violated because of the uncertainty principle exactly as it is in the fast process of coherence appearance.

3. Ideal Bose gas

In this section we consider the simplest of possible models—a collisionless Bose gas which initially is at a temperature $T_0 > T_c$. This gas is being cooled by the heat bath at a temperature $T_F < T_c$. The bath is assumed to have an infinite heat capacity. From thermodynamic considerations we know that Bose condensation must occur in the system if the number of particles is kept constant. Thus, it is our task to follow the time evolution of the distribution function of bosons from the initial equilibrium state:

$$n(\omega, 0) = \left[\exp\left(\frac{\hbar\omega + \mu}{T}\right) - 1\right]^{-1} \tag{3.1}$$

to the final state

$$n(\omega, t \to \infty) = \left[1 - \left(\frac{T_F}{T_c}\right)^{3/2}\right](2m)^{3/2}\frac{\delta(\omega)}{\omega^{1/2}} + \frac{1}{e^{\hbar\omega/T} - 1}. \tag{3.2}$$

To do it we must solve the Boltzmann equation (Levich and Yakhot 1977a, b):

$$\frac{\partial n_K}{\partial t} = I_{BT} \tag{3.3}$$

The solution must conserve the total number of particles:

$$\int n(\omega)\omega^{1/2}\,d\omega = N_0 = \text{constant}. \tag{3.4}$$

The approximate procedure leads to a simplified equation for $n(\omega, t)$ (Levich and Yakhot 1977a, b):

$$\frac{\partial n(\omega, t)}{\partial t} = (-\alpha\omega^{1/2} + \beta F(t))n(\omega, t) - \gamma n(\omega, t) + \delta f(\omega)\frac{T}{\omega^{1/2}} \tag{3.5}$$

where $F(t)$ is an unknown functional of $n(\omega, t)$ which ensures that (3.5) conserves the total number of particles:

$$F(t) \propto \int \omega n(\omega, t)\,d\omega; \qquad f(\omega) = \begin{cases} 1 & 0 < \hbar\omega < \mu_F \\ 0 & \hbar\omega > \mu_F \end{cases}$$

and all the other notations can be found in Levich and Yakhot (1977b). We can now

write the solution of (3.5) and find the unknown functional $F(t)$ from the normalisation condition (3.4). For the large values of time t one obtains:

$$n(\omega, t) \simeq N_0 \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right] t^3 \exp(-\alpha \omega^{1/2} t) + \frac{T}{\hbar \omega} \tag{3.6}$$

at $\hbar\omega \ll T$.

This solution is obtained for initial condition (3.2) and leads to:

$$n(K, t) = N_0 \left[1 - \left(\frac{T}{T_c} \right)^{3/2} \right] \delta(\boldsymbol{K}) + \frac{T}{\hbar \omega}$$

when $t \to \infty$. One can check easily that during the evolution condition (3.4) is fulfilled. If at $t = 0$ there is a finite fraction of particles already in condensate:

$$n(\omega, 0) = A \frac{\delta(\omega)}{\omega^{1/2}} + \frac{1}{e^{\hbar\omega/T_0} - 1}$$

the temporal evolution at large times is given by

$$n(E, t) \simeq \left[-1 \left(\frac{T}{T_c} \right)^{3/2} \right] \frac{A\alpha t \delta(\omega)/\omega^{1/2}}{A\alpha t + T_0 - T} + B(\omega, t) \tag{3.7}$$

where $B(\omega, t)$ describes fast relaxation of particles with large $\boldsymbol{K}$ to the Planck distribution.

One can notice the important difference between (3.6) and (3.7). Expression (3.6) describes a process leading to a δ-function formation only when $t \to \infty$. This means that the condensed phase cannot be formed. On the other hand (3.7) gives an absolutely different type of behaviour: the δ-function part of the distribution is always present but the mass of the condensed phase changes with time reaching the equilibrium magnitude $m = 1 - (T/T_c)^{3/2}$ at $t \gg (T_0 - T)/A\alpha$. We can conclude that with the presence of nuclei of a new phase the transition takes a finite time while without nuclei the time is infinite.

We would like to point out that the main feature of the dynamics of Bose condensation given by (3.6) is the growth of small momenta modes simultaneously with the width of the region in which growth occurs being diminished. This is the feature which does not exist in linear relaxational dynamics at all, and which makes the problem considered in this paper very different in comparison with quasi-equilibrium dynamics. Let us end our discussion of the ideal Bose gas with two comments which will prove to be important for further more complicated models:

(1) If the final temperature is close to the critical temperature, $|T_F - T_c| \ll T_c$, then only small fraction of the particles is required to form the Bose condensate. In the ideal Bose gas this fraction is proportional to $|T_F - T_c| \simeq 1/\xi$, where ξ is the correlation length. Thus we may argue that initially during the linear stage of evolution only the modes $n(K < 1/\xi)$ begin to increase while all others decrease. When $n(K < 1/\xi)$ is well beyond an equilibrium value the evolution of these modes proceeds in accordance with the theory developed above. We will see that this comment is also of value for realistic models with interaction in the critical region.

(2) The evolution given by (3.6) is unfortunately not always valid.

It is easy to understand that when the width of the peak ΔE_0 is so narrow that condition (2.9), $t > \hbar/\Delta E_0$, is no longer fulfilled we cannot use the golden rule which leads us to the Boltzmann equation. In other words, the system acquires coherence

which demands to be taken care of as was described in the previous section. It is remarkable that the Boltzmann equation which is, strictly speaking, not suitable to describe non-random systems has so much dynamics inherent in it that it still gives us the formal solution in the form of the appearance of a δ-function.

4. Weakly interacting Bose gas: far from the critical temperature

We now consider a system of weakly interacting Bose particles with the initial and final temperatures both far from the critical temperature: the system at a temperature above T_c at $t=0$ is cooled to a temperature T, well below T_c. Thus, at the present stage, we are not concerned with critical fluctuations and critical behaviour in the usual sense. In some respects this situation is similar to that investigated by Kawasaki *et al* (1978) when they tried to follow the temporal evolution of a system instantaneously quenched to a temperature far below the spinodal decompositon point.

To follow the appearance of the new phase we must solve equation (2.14). It is obviously a hopeless task to do it exactly and thus we will make certain simplifications based on physical considerations.

At the initial moment, the collision integral I_{BB} corresponding to interaction between Bose particles is zero while the equilibrium with the heat bath is already broken ($I_{BT} \neq 0$). We may conclude that the initial stages of relaxation can be described exactly in the way that was done for an ideal Bose gas until $I_{BB} \ll I_{BT}$. If the perturbation is strong enough this relaxation leads to a considerable increase in the boson occupation numbers at small momenta with the subsequent appearance of a low energy peak in the distribution function.

When this peak is developed I_{BB} plays the dominant role in the time evolution because it contains the induced scattering terms which are proportional to n_K^3 while the Bose particle–heat bath interaction contains n_K^2 only. From now on we can neglect $|I_{BT}| \ll |I_{BB}|$ and consider the evolution which is governed by I_{BB} only. We make an additional assumption that the initial perturbation was so strong that this peak contains all the would-be condensate particles. Thus the distribution function consists of two parts: the high energy equilibrium Planck tail and low energy peak which tends to form a δ function. The high energy part of the system serves as an additional heat bath for the relatively slowly developing condensed phase and its existence must be taken into account. It should be emphasised that this peak contains only the excess of particles which cannot be included in the Planck part of the distribution at a given temperature. Those particles have only one way to relax to equilibrium, i.e. to form a Bose condensate. Thus it is reasonable not to consider the processes which lead to the broadening of the peak but to leave only those terms of I_{BB} leading to narrowing of the peak and its movement in the low energy direction. Thus we consider an equation

$$\frac{\partial n_{\boldsymbol{K}}}{\partial t} = -\frac{U_0^2}{\hbar} \sum \delta(\Delta \boldsymbol{K}) \delta(\Delta\omega) n_{\boldsymbol{K}_1} n_{\boldsymbol{K}_1'} (n_{\boldsymbol{K}_2} - n_{\boldsymbol{K}_2'}) \tag{4.1}$$

where $\delta(\Delta \boldsymbol{K})$ is the Kronecker symbol. We have omitted the terms

$$\sum \delta(\Delta \boldsymbol{K}) \delta(\Delta\omega) n_{\boldsymbol{K}_2} n_{\boldsymbol{K}_2'} (n_{\boldsymbol{K}} - n_{\boldsymbol{K}_1}) \simeq \sum n_{\boldsymbol{K}_2} n_{\boldsymbol{K}_2'} \frac{\partial^2 n}{\partial \omega^2} (\omega_2 - \omega_4)^2 = D \frac{\partial^2 n}{\partial \omega^2}$$

with $D > 0$.

We see easily that this term describes the process of the peak broadening and is thus omitted. It is clear that the first-order expansion porportional to $(\omega_2-\omega_4)$ in the 'diffusion term' is equal to zero, because of its antisymmetric properties. Applying an assumption that the energy exchange per collision is small $\Delta\omega \ll \omega$ and using techniques described elsewhere we arrive at a simplified equation (Levich and Yakhot 1977b)

$$\frac{\partial n_K}{\partial t} = B \int n_{\boldsymbol{K}} n_{\boldsymbol{K}'} (\omega^{1/2} - \omega'^{1/2}) \psi(t)\, \mathrm{d}^3\boldsymbol{K}' \tag{4.2}$$

where B is a constant and

$$\psi(t) = \int \frac{\partial n(k_x, k_y, 0)}{\partial \omega}\, \mathrm{d}k_x\, \mathrm{d}k_y. \tag{4.3}$$

This equation conserves the total number of particles in the peak. We must admit that the energy conservation law is violated by our approximations but we believe that it is not of importance because the energy of this low energy part of the system is very small, tending to zero.

The solution of (4.2) is:

$$n \approx \frac{N_0 n(\omega, 0)}{m^{3/2}} \frac{\exp(-B\omega^{1/2} F(t))}{\int n(\omega, 0) \exp(-B\omega^{1/2} F(t))\, \mathrm{d}\omega} \tag{4.4}$$

where

$$N_0 \approx N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] = \int n\, \mathrm{d}^3\boldsymbol{K} \qquad \text{and} \qquad F(t) = \int_0^t \psi(x)\, \mathrm{d}x.$$

This solution consreves the total number of particles and leads to a δ-function form if $F(t)$ increases with time, being positive. To show this let us use the simple intiial condition

$$n(\omega, 0) \propto \exp[-(\omega/\omega_0)^{3/2}]. \tag{4.5}$$

It can be shown that the results do not depend on the shape of $n(\omega, 0)$ at large times and we use (4.5) as one of the possible initial conditions. Expression (4.4) with (4.5) must be inserted into (4.3) and the function $F(t)$ can be found without difficulties. The result is

$$n \approx \frac{\hbar^3 N_0}{m^{3/2}} \frac{\exp[-\omega^{1/2}/(\omega_0 - ct)^{1/2}]}{(\omega_0 - ct)^{3/2}} \tag{4.6}$$

where $c = N_0^2 U_0^2 / 2\hbar^2 V^2$. Thus at

$$t_0 = \frac{2\hbar^2 \omega_0 V^2}{N_0^2 U_0^2} \tag{4.7}$$

the solution is:

$$n(\omega, t_0) = \frac{N_0}{m^{3/2}} \frac{\delta(\omega)}{\omega^{1/2}}. \tag{4.8}$$

We can see that the Boltzmann equation can lead to a δ-function form or, in other words, to the appearance of coherence in the system. This fact is quite remarkable since we have explained above that in the final stages of the process when the peak is

narrow the Boltzmann equation does not work at all since all coherence effects are removed during its derivation. In fact the condition of validity for the Boltzmann equation is the following:

$$t_0 \simeq \frac{2\hbar^2\omega_0 V^2}{N_0^2 U_0^2} > \frac{\hbar}{E_0} \simeq \tau_{\text{coherence}}. \tag{4.9}$$

When this condition is not fulfilled we should use the appropriate equation of motion.

We finish this section with a comment about an explosive solution we obtained above. Unfortunately, we cannot judge rigorously whether this explosive character of the solution is an artefact of our approximation or the result of a strong non-linearity implied in the equation. In principle explosive solutions of this type are well known in plasma physics and occur when the instability is strong.†

5. The evolution of the coherence stage of the peak

To proceed further with this investigation of the fate of the peaks we pass over now to the approximate equation (2.16).

To begin with, it is easy to check that $n_{\boldsymbol{K}} = \delta(\boldsymbol{K})$ is the stationary solution of this equation. We will be looking for the solution of equation (2.16) which leads to δ-function form with time and which describes the fast relaxation:

$$|\boldsymbol{K}_1^2 - \boldsymbol{K}_3^2 + \boldsymbol{K}_2(\boldsymbol{K}_3 - \boldsymbol{K}_1)|t \ll 1. \tag{5.1}$$

This condition is the same as (2.15) in which it is assumed that the energy transfer per collision is small and thus the term proportional to $|\Delta\boldsymbol{K}|^2$ is omitted. With this assumption and using the physical arguments discussed above, we can write:

$$\frac{\partial n_{\boldsymbol{K}}}{\partial t} = -\frac{V_0^2 t}{\hbar^2 V^2} \sum n_{\boldsymbol{K}} n_{\boldsymbol{K}_3} \frac{\partial n_{\boldsymbol{K}}}{\partial \boldsymbol{K}_2} \frac{\boldsymbol{K}_2}{\boldsymbol{K}_2} (\boldsymbol{K}_1 - \boldsymbol{K}_3). \tag{5.2}$$

It is possible to solve equation (5.2) taking into account restriction (5.1) in the form:

$$\boldsymbol{K}_2(\boldsymbol{K}_3 - \boldsymbol{K}_1) \simeq -\boldsymbol{K}_1^2 + \boldsymbol{K}_3^2. \tag{5.3}$$

Substituting (5.3) into (5.2) we have finally:

$$\frac{\partial n_{\boldsymbol{K}}}{\partial \tau} = n_{\boldsymbol{K}} \phi(t) \sum (\boldsymbol{K}_3^2 - \boldsymbol{K}_1^2) n_{\boldsymbol{K}_3} \tag{5.4}$$

† *Note added in proof.* We would like to stress that in all derivations shown above it is assumed that the low energy peak contains all would-be condensate particles. This is true if the perturbation is strong enough. In fact we assume that already at the first stages of evolution, when $I_{BB} < I_{BT}$ the peak is so well developed that it contains $N_0 \propto \left(1 - \frac{T}{T_c}\right)^{3/2}$ particles. After this peak has been formed and is narrow enough, the evolution is accelerated by the induced scattering process between the Bose particles. The term which tends to destroy this peak is presumably weaker, because as we see from (4.1) it is equal to

$$D\frac{\partial^2 n}{\partial \omega^2} \text{ with } D \propto K; \quad D \to 0, \quad K \to 0$$

If the perturbation is not too strong the other situation may occur. The peak will not be well developed and will contain an amount of particles smaller than N_0. In this case less then equilibrium number of particles might appear in the condensate. This corresponds effectively to the temperature higher than T_F. The further relaxation to T_F might be of quasi-equilibrium nature described by (3.7).

where $\tau = U_0^2 t^2 / 2\hbar^2 V^2$ and

$$\phi(t) = -\sum \frac{\partial n_{\boldsymbol{K}}}{\partial \boldsymbol{K}} \frac{1}{\boldsymbol{K}}. \tag{5.5}$$

Equation (5.4) conserves the total number of particles in the peak and the solution can be written readily:

$$n_{\boldsymbol{K}} = \frac{N_0 n_{\boldsymbol{K}}(0) \exp(-\boldsymbol{K}^2 F(\tau))}{\int n_{\boldsymbol{K}}(0) \exp(-\boldsymbol{K}^2 F(\tau)) \boldsymbol{K}^2 \, \mathrm{d}\boldsymbol{K}} \tag{5.6}$$

with $F(\tau) = \int_0^\tau \phi(X)\,\mathrm{d}X$. Based on the results of previous sections we assume that $n_{\boldsymbol{K}}(0) \to \text{constant} > 0$ at $\boldsymbol{K} \to 0$ and for $t \to \infty$ we can write if $F(t)$ grows being positive (this will be checked further)

$$n_{\boldsymbol{K}} \propto F^{3/2}(\tau) \exp(-\boldsymbol{K}^2 F(\tau)). \tag{5.7}$$

Recalling the definition (5.5) we obtain the following equation for $F(\tau)$:

$$\frac{\mathrm{d}F}{\mathrm{d}\tau} = \phi(\tau) \propto F^{5/2} \int \exp(-\boldsymbol{K}^2 F) \boldsymbol{K} \, \mathrm{d}\boldsymbol{K} \tag{5.8}$$

and consequently:

$$F(\tau) = \frac{1}{\boldsymbol{K}_0} \mathrm{e}^{\tau} \tag{5.9}$$

where κ_0 is the dimensional constant. Thus

$$n_{\boldsymbol{K}} \propto \mathrm{e}^{\frac{3}{2}\tau} \, \mathrm{e}^{-\boldsymbol{K}^2 \mathrm{e}^{\tau}}$$

which leads to a δ-function at $t \to \infty$. The characteristic time (5.5) is the same as that derived above on the basis of dimensionality considerations.

The main drawback of the derivation given in this section is that it does not take into account the changes of the energy spectrum in the system which accompany the appearance of coherence. This problem will be treated in the next section while here we would like to discuss once more the procedure we have been using till now.

The Boltzmann equation (2.16) describes a balance between transitions per second from and to a state with given momentum $\boldsymbol{K}$. This equation is written in a second-order of perturbation theory and it is easy to understand that coefficients of $\sin^2(\Delta\omega t)/[(\Delta\omega)^2 t]$ correspond to the non-diagonal matrix elements of the interaction operator.

We can readily conclude that these transitions are not the only effect of interaction. The frequencies of the modes are changed and their shifts can be estimated if one calculates the diagonal matrix elements of this interaction.

6. Extreme coherence in evolution: formation of a new phase

We would like to emphasise that comparatively simple ways of getting a solution for the complicated non-linear equations we deal with are possible only because the evolution which we are most interested in occurs in a narrow interval in momentum space. Similar situations are well known in different branches of non-linear physics: in optics, in plasma physics, etc. There the propagation and development of a narrow

wave packet are thoroughly investigated and well understood (Zakharov 1972). In fact we use very similar techniques and methods with respect to non-linear dynamics of phase transitions. To illustrate the similarity let us consider the simple situation when we have only four modes in the system interacting with each other. Then, as is well known, if the frequencies of these four modes satisfy the resonance condition $\omega_1+\omega_2=\omega_3+\omega_4$ the intensive enhancement of one or two modes at the expense of the others may occur. For example, the amplitude of modes 1, 3 may rise until 2 and 4 shrink to zero. Afterwards the process may be reversed. The situation is exactly analogous to that of parametric resonance. If the resonance condition is not fulfilled there is no parametric resonance because of the beats which effectively damp the growth of amplitudes. One should not mix together the resonance condition with the conservation of energy in the microscopic act of collision between the waves. On the contrary the resonance approximation is opposite to the approximation when the energy is conserved during the act of collision. From the general quantum mechanical point of view it is clear that the wave's frequency is changed due to non-linear waves interacting.

If the system contains an infinite number of strongly coupled modes the situation is profoundly more complex. The evolution in this case is governed by the system of Dyson–Wild equations (Zakharov and Lvov 1975). However only in simple cases are these equations really useful. The first case is an approximation of weak turbulence which we have discussed above. The second one is when the spectral width of excitations is small. One can easily show that the δ function is an exact stationary solution for this system. Though it is difficult to prove that this stationary solution is indeed achieved, it is logical to assume, physical considerations, that the simplest stable solution is to be chosen. Our argument is, however, even milder than that. We assume that in a narrow band of modes, strongly coupled with each other, complete coherence is achieved on the time scale of dynamical interaction. Afterwards we are able to apply the envelope approximation, i.e. approximation in which one neglects the fine structure of interaction within the spectral band and considers only the envelope motion. The time scale of dynamic interaction is given by (2.17).

For the details of the envelope approximation we refer to work by Zakharov (1972). In this approach one encounters a variety of non-linear phenomena such as self-focusing, instability in coordinate space, etc. With respect to our problem we will show, however, that no such spectacular pecularities occur. In the envelope approximation the Hamiltonian equation of motion for the very narrow peak is given by (see Zakharov 1972)

$$\left(\frac{\partial}{\partial t}+i\tilde{\omega}_{k_0}(r)+v_{k_0}\frac{\partial}{\partial r}+\omega'_{k_0}\nabla_r^2\right)a_k(r)=0 \tag{6.1}$$

there k_0 is the central momentum of the peak which tends to 0 with time,

$$v_{k_0}=\left.\frac{\partial\omega}{\partial k}\right|_{k=k_0};\qquad \omega''=\left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=k_0} \tag{6.2}$$

$$\tilde{\omega}_k=\omega_k+\frac{U_0N_0}{\hbar V}\qquad\text{and}\qquad\langle a_k^{\dagger}a_k\rangle=N_0. \tag{6.3}$$

Generally speaking v_k, ω_k'' and the amplitude are now functions of coordinates which reflects the spatial coherence in the system. The meaning of equation (6.1) is that we express the interaction in the exact dynamic equation in the form of a renormalised

frequency $\tilde{\omega}_k$. On the other hand in (6.1) we neglect collisions. In the sense the approximation of (6.1) is similar to the mean-field approximation in the usual Boltzmann equations, though it corresponds to the completely opposite case of a coherent rather than a random system.

Equation (6.1) has a simple homoheneous solution

$$a = a_0 \exp(-i\tilde{\omega}_{k_0} t) \tag{6.4}$$

which represents a homogeneous wave propagating with its frequency altered as a result of interaction. Since $\omega_k \to 0$ the altered frequency should give us the energy of the ground state of Bose condensate. It is easy to see that the value of the energy shift coincides exactly with the corresponding expression of Bogolubov for the ground state energy of a weakly interacting Bose gas. The time during which the process takes place is again determined either by the uncertainty principle as $\hbar V/U_0 N_0$ or from the quasi-Boltzmann equation as was done in the previous section. The results are obviously identical. However, this is not the end of the story. Indeed we have not proved that the homogeneous solution is the stable one with respect to non-homogeneous perturbations. If we look for the solution of (6.1) in a form

$$a = a_0 e^{i\tilde{\omega}t} + \psi(r, t); \qquad |\psi| \ll |a_0| \tag{6.5}$$

where

$$\psi(r, t) \propto e^{i\Omega t - iq.r} \tag{6.6}$$

we will easily derive the dispersion relation for $\Omega(q)$ (Lvov and Rubenchik 1977):

$$\Omega(q) = \boldsymbol{q} \cdot \boldsymbol{k}_0 \pm q\left(\frac{q^2}{4m^2} + \frac{2U_0 N_0}{V}\right)^{1/2}. \tag{6.7}$$

When k_0 goes to zero, i.e. the ground state is formed, equation (5.7) coincides with well known Lee, Yang and Huang spectrum of excitations.

The conclusions we draw are the following. The homogeneous solution corresponding to a very narrow peak of non-equilibrium states is stable in *coordinate space*. Being unstable in the *phase space* it forms an exact Bose coordinate, i.e. ground state, during the time given above. The spectrum of excitations which appears as a result of perturbation of the homogeneous solution is the usual phonon spectrum, when the width of the peak tends to zero.

7. The critical region

In § 2 we have derived the dynamic equation (2.24) appropriate for the critical region and valid for modes $K < 1/\xi$ which can even be in a strongly non-equilibrium state as well. If the final temperature is within the critical region then in the equilibrium final state the superfluid fraction is proportional to the correlation length $\xi \sim 1/|T - T_c|^{\nu}$ exactly as in an ideal Bose gas. Therefore we may expect that only modes which become non-equilibrium states and are unstable in the momentum-space peak also have $K < 1/\xi$. But for these nodes equation (2.24) is valid whether they are close to or far from equilibrium.

The only complication arises from the fact that now the renormalised vertex consists of two parts: the first is the constant which is proportional to ϵ and the second

is K dependent:†

$$\Gamma = \tilde{U}_0 + \frac{DK^2}{-i\omega + DK^2}\tilde{U}_0. \tag{7.1}$$

The imaginary part of Γ brings new features to the kinetic equation which can be derived from (2.24). It is easy to check that:

$$\frac{\partial n_K}{\partial t} = -DK^2 n_K + \frac{\tilde{U}_0 D n_K}{\hbar V}\sum \frac{(\omega_K - \omega_{K'})(K-K')^2}{(\omega_K - \omega_{K'})^2 + D^2(K-K')^4} n_{K'} + \tilde{I}_{BB} \tag{7.2}$$

in which I_{BB} is proportional to $(\mathrm{Re}\,\Gamma)^2$:

$$\mathrm{Re}\,\Gamma = \tilde{U}_0\left(1 + \frac{D^2K^4}{\omega^2 + D^2K^4}\right). \tag{7.3}$$

In the critical region where $D \to \infty$ we can neglect the K-dependence of $\mathrm{Re}\,\Gamma$. The same occurs when we are interested in the narrow region of K near to $|K| = 0$. It is evident that in this case $|K'^2 - K^2| \ll (K-K')^2$ and we can neglect $\omega_K - \omega_{K'}$ in (7.2) and (7.3) in comparison with DK^2. We have finally:

$$\frac{\partial n_K}{\partial t} = -DK^2 n_K + \frac{\tilde{U}_0}{V\hbar} n_K \sum \frac{\omega_K - \omega_{K'}}{D(K-K')^2} n_K + \tilde{I}_{BB} \tag{7.4}$$

and now $I_{BB} \propto \epsilon^2$.

In addition to a new diffusion term which is linear equation (7.4) also contains the non-linear (apart from $\tilde{I}_{BB}$) contribution from the integrated out modes with large $|K|$. We have pointed out that these modes can be treated as an additional heat bath and this can lead to Bose condensate formation even if $\tilde{I}_{BB} = 0$ as in the case of an ideal Bose gas discussed in § 2. We are not going to discuss the role of these processes in the course of Bose condensation because at the final stages of relaxation, when $n_K \gg 1$, the condensation is governed by I_{BB} which is the largest term in equation (7.4). Thus we can conclude that the most interesting stages of the phase transition in the critical region are in principle the same as those far from T_c.‡

8. Conclusions

Wave studied the dynamics of Bose condensation in an interacting Bose gas. It is shown that:

(1) The dynamics of this phase transition is essentially a non-equilibrium phenomenon. The overall picture of the time evolution of the formation of the new phase is the following. During the first stages of relaxation the particle–heat bath interaction plays a most important role. The system is still random and thus the

† *Note added in proof.* In fact there should be several K dependent terms in (7.1) due to the different hydrodynamic modes. However, for our purposes it is sufficient to use only one typical term.

‡ *Note added in proof.* We might consider as a time of Bose condensation the time during which coherence is achieved within the narrow band. This time is proportional to $(1/N_0)\alpha|T - T_c|^{-\nu}$, so that here we also indicate by N_0 the critical slowing down.

One can see that both in the critical region and outside it the very last stage of relaxation equilibrium is governed by the second term in (7.4). Physically the equilibrium is finally achieved due to the transfer of energy within the coherent band to the 'heat bath', the short wavelength part of the system. In the critical region this time is certainly much larger than the time scale of dynamic interaction.

Boltzmann equation can be used. At this stage of relaxation the peak of the distribution function is created in the small-K region. The maximum of this peak grows linearly with time: $n_{max} \propto t$ while the width of this peak decreases.

After this peak is high enough the most important part of the interaction which governs relaxation is the non-linear interaction between the Bose particles. At those stages in which coherence is not yet strongly developed one can use the Boltzmann equation. The peak growth becomes of explosive character.

When this peak is too narrow and coherence starts to be developed in the system, the Boltzmann equation is no longer valid†.

(2) We have shown how to derive the Halperin–Hohenberg–Siggia macroscopic equations for liquid helium using purely microscopic considerations.

(3) The smooth and gradual (in time) appearance of coherence properties in the initially random system is the crucial stage in the theory. The profound analogies with various non-linear problems in optics, physics of plasmas and hydrodynamics are understood.

(4) We expect that the seond-order phase transitions in spin systems are dynamically similar to Bose condensation considered in this paper.

† *Note added in proof.* Qualitatively we may use the 'reversible' quasi-Boltzmann equation described above. However it is important to realize that the system in the course of phase transition passes through a stage which may be identified as a period of strong turbulence. This period is followed by the stage of developed coherence. The phonon type spectrum of excitations is obtained by considering the equations of motion in the envelope approximation. In the critical region the time of Bose condensation divergents as $|T - T_c|^{-\nu}$. This time is determined by the reversible mode-mode coupling terms in the equation of motion. The final stage of relaxation to equilibrium is determined by disspitative interaction between the coherent and stochastic modes in the system.

References

Chaplin G, Cooper G and Slutz S 1974 *Phys. Rev.* A **9** 1273
Coste J and Peyrand J 1975 *Phys. Rev.* A **12** 2144
Goldhirsch I, Levich E and Yakhot V 1978 to be published
Halperin B I, Hohenberg P C and Ma S K 1974 *Phys. Rev.* B **10** 137
Halperin B I, Hohenberg P C and Siggia E 1976 *Phys. Rev.* B **13** 1299
Hohenberg P C and Halperin B I 1977 *Rev. Mod. Phys.* **49**
Kawasaki K 1976 *Phase Transitions and Critical Phenomena* vol. 5a, eds C Domb and M S Green (New York: Academic)
—— 1977 *Prog. Theor. Phys.* **57** 410
Kawasaki K, Yalabik M C and Gunton J D 1978 *Phys. Rev.* A **17** 455
Klander J R 1960 *Ann. Phys., NY* **11** 123
Kwok P C and Martin P C 1966 *Phys. Rev.* **142** 495
Langer J S 1968 *Phys. Rev.* **167** 183
—— 1969 *Phys. Rev.* **184** 219
Levich E and Yakhot V 1977 *Phys. Rev.* B **15** 243
—— 1977b *J. Low Temp. Phys.* **28** 197
Lvov V S and Rubenchik A M 1977 *Sov. Phys.-JETP* **72** 127
Ma S K and Senbetu L 1977 *Phys. Rev.* A **10** 2401
Marris H J 1964 *Phil. Mag.* **9** 901
Mori H 1965 *Prog. Theor. Phys.* **33** 423
Sak J 1976 *Phys. Rev.* B **14** 2932
Suzuki M 1976 *Prog. Theor. Phys.* **56** 77
—— 1977 *Prog. Theor. Phys.* **57** 380
Sagdeev R 1973 *Nonlinear Plasma Theory* (Moscow: Atomisdat)
Zakharov V E 1972 *Sov. Phys.-JETP* **62** 1743
Zacharov V E and Lvov V S 1975 *Isvestiya Vuzov, Radiofizilau* **10**
Zeldovich Ya B and Levich E 1969 *Sov. Phys.-JETP* **28** 1287